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Thermo-fluid dynamics of the unsteady channel flow

Amilcare Pozzi, Renato Tognaccini *

Dipartimento di Ingegneria Aerospaziale, Università di Napoli "Federico II", Naples, Italy

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ABSTRACT

We present an exact analytical representation of the unsteady thermo-fluid dynamic field arising in a two-dimensional channel with parallel walls for a fluid with constant properties. We assume that the axial pressure gradient is an arbitrary function of time that can be expanded in Taylor series; a particular case is the impulsive motion generated by a sudden jump to a constant value; for large time values the flow reaches the well-known steady Poiseuille solution. As boundary conditions for the dynamic field we consider fixed and moving walls (unsteady Couette flow). The assigned temperature on the walls can be an arbitrary function of time. We also consider the coupling of the energy and momentum equations (i.e. Eckert number different from zero). The solution is obtained by series with simple expressions of the coefficients in terms of the error functions. The fundamental physical parameters, such as shear stress, mass flow and heat flux at the wall are obtained in explicit analytical form and discussed by means of their diagrams.

1. Introduction

Unsteady laminar flows in pipes have been widely studied since the classical paper of Szymanski [1], in which the author proposed the exact analytical solution describing the incompressible, laminar flow arising in a circular pipe initially at rest and accelerated by a sudden jump of the pressure gradient to a constant value. This is among the few exact analytical solutions of the unsteady Navier–Stokes equations.

A renewed interest in the analytical analysis of these problems can be found in literature. The recent book of Drazin and Riley [2] provides a review of the available exact solutions of the Navier–Stokes equations. Ingham and Pop [3] focus their interest on solutions of the energy equation. In the case of unsteady flows interesting results have been presented in [4] with the analysis of the heat transfer in impulsive Falkner–Skan flows.

Even if limited to simple geometries, analytical solutions can provide a more clear understanding of the influence of the parameters governing the physical problem, can be used for a quantitative analysis of more complex geometries looking at their asymptotic behavior and are very useful test cases for more general numerical methods. Nonetheless, studies devoted to the analysis of two-dimensional channel flows are only a few. It can be due, as discussed in [5], "... possibly because of the difficulties cited by Trikha [6] in deducing inverse Laplace transform for all but a few simple analytical functions". In the same paper, Brereton proposed the analysis of fully developed laminar flows in pipes and channels for arbitrary unsteadiness, prescribing the time law of the mass flow rate instead of the pressure jump considered by Szymanski. The analytical solution was found explicitly in the Laplace transformed variables; the inverse transform could be obtained by the method of residual and was performed numerically to derive section properties as pressure gradient and shear stress on the wall.

Das and Arakeri [7] proposed analytical inverse Laplace transforms for some particular flows in pipes and channels again with prescribed mass flow rate in time. The cases of trapezoidal law and impulsive acceleration to a constant mass flow rate were studied. Extensions and applications to the case of non-Newtonian fluid can be found in [8].

More recently Brereton and Jiang [9] studied the thermal field associated to the laminar flows in pipes and channels with assigned axial temperature gradient in the case of negligible effects of the dissipation of kinetic energy, i.e. with zero Eckert number.

Indeed, the problem is strongly complicated when the thermal field is coupled with the velocity field: no exact solution including the effects of the dissipation of kinetic energy (Eckert number different from zero) are available in literature.

In this paper an exact explicit analytical solution for both the velocity and thermal field including the effect of the Eckert number is presented: the incompressible, laminar, fully developed flow of a fluid accelerated from rest in a two-dimensional channel with flat, parallel walls, by assigning a general time law for the pressure gradient (unsteady Poiseuille flow). This solution is also extended to the case in which the motion of the fluid, initially at rest, is generated prescribing a general time law for the velocity of the

* Corresponding author. Dipartimento di Ingegneria Aerospaziale, Universitá di Napoli "Federico II", Piazzale V. Tecchio 80, 80125 Napoli, Italy. Tel.: +39 081 768 2179; Fax: +39 081 768 2187.

E-mail address: renato.tognaccini@unina.it (R. Tognaccini).

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upper wall in the direction of the channel axis (unsteady Couette flow). For the thermal field a time varying constant wall temperature along the channel is assumed.

The solution has been found by the Laplace transform technique. The inverse Laplace transforms have been obtained using some interesting properties of the Jacobi's θ_2 and θ_3 functions leading to series in terms of the well-known error function rapidly converging.

The presence of a not simple non-homogeneous term in the energy equation makes its integration very difficult when the Eckert number is taken into account. However, in the present paper a simple analytical expression of a particular integral of the non-homogeneous equation has been found that enables to easily obtain an analytical solution in the case of Prandtl number equal to one. Therefore, explicit relations for the axial velocity and temperature profiles across the channel, shear stress and Nusselt number have been derived.

The local analysis for small time values, reveals the existence of two regions in the flow: near the wall a self-similar *Rayleigh type layer*, far from the wall a *potential region*, where the velocity is constant across the channel, zero in the case of the unsteady Couette flow and proportional to a power of the non-dimensional time for the unsteady Poiseuille flow.

The method of solution can be easily applied when the time law imposed for the pressure gradient or upper wall velocity is of polynomial type and a number of applications are proposed with the analysis of the effects of the involved parameters.

2. Governing equations

We consider an incompressible, laminar flow of a fluid with constant properties in a two-dimensional plane channel with parallel flat walls. The non-dimensional spatial coordinate orthogonal to the wall Y is referenced to the semi-height d of the channel, with the origin placed on the channel axis.

The velocity field is governed by the Navier–Stokes equations. In the case of a steady flow with constant pressure gradient along the axis (dp/dx), the solution is the well-known Poiseuille flow: the velocity field is parallel, independent of the position along the axis with a parabolic distribution across the channel of the axial velocity component.

A time varying law for the pressure gradient is imposed along the channel axis. We specify with u and v respectively the non-dimensional velocity components aligned and orthogonal to the channel axis, adopting as reference velocity $V_{\mathrm{ref}} = -d^2(dp/dx)_{\mathrm{ref}}/(2\mu)$ (μ is the dynamic viscosity). Again, being equations independent of x, $v = 0$ due to the continuity equation. Denoting with τ the non-dimensional time referenced to $t_{\mathrm{ref}} = d^2/\nu$ (ν is the kinematic viscosity), the momentum balance equation along the x axis reduces to

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial Y^2} = 2\, r(\tau), \tag{1}$$

where $r(\tau)$ is the imposed non-dimensional pressure gradient.

The initial and boundary conditions for this, second order, linear parabolic equation that we assume are

$$u(Y,0) = 0, \tag{2a}$$

$$u(\pm 1,\tau) = 0; \tag{2b}$$

the latter providing the no-slip condition on the walls.

Denoting with $\theta = (T - T_0)/T_0$ the non-dimensional temperature of the fluid, where T_0 is the constant temperature at $\tau = 0$ and considering problems in which the boundary conditions for the temperature are independent of the x-direction, the energy equation for a fluid with constant properties is:

$$\frac{\partial \theta}{\partial \tau} - \frac{1}{Pr}\frac{\partial^2 \theta}{\partial Y^2} = E\left(\frac{\partial u}{\partial Y}\right)^2, \tag{3}$$

where $Pr = \frac{\mu c_p}{\lambda}$ specifies the Prandtl number and $E = V_{\mathrm{ref}}^2/(c_p T_0)$ is the Eckert number we defined for the present problem (c_p is the specific heat and λ the thermal conductivity of the fluid).

In this case (solution independent of x) the temperature is connected with the velocity only through the dissipation of kinetic energy. The initial and boundary conditions we assume are

$$\theta(Y,0) = 0, \tag{4a}$$

$$\theta(+1,\tau) = \theta_w^+(\tau), \tag{4b}$$

$$\theta(-1,\tau) = \theta_w^-(\tau); \tag{4c}$$

where $\theta_w^+(\tau)$ and $\theta_w^-(\tau)$ respectively specify the imposed temperature on the upper and lower wall of the channel.

In this case of constant properties the dynamic field is not coupled with the thermal field and can be independently solved. Moreover, if $E = 0$, velocity and temperature fields are completely independent. Both the momentum and the energy equations are linear and the associated homogeneous equation is the same: the well-known parabolic heat equation.

3. The method of solution

The problem can be solved by the Laplace transform technique. The solution of the associated homogeneous problem, for both momentum and energy equations, can be found by the same procedure and has the same structure.

The general problem consists in the solution of the equation

$$\frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial Y^2} = g(Y,\tau), \tag{5}$$

where $g(Y,\tau) = 2\, r(\tau)$ for the momentum equation and $g(Y,\tau) = E(\partial u/\partial Y)^2$ for the energy equation. If $g(Y,\tau) = 0$, this equation reduces to the homogeneous heat equation:

$$\frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial Y^2} = 0. \tag{6}$$

The initial and boundary conditions are:

$$f(Y,0) = 0, \tag{7a}$$

$$f(+1,\tau) = f_w^+(\tau), \tag{7b}$$

$$f(-1,\tau) = f_w^-(\tau). \tag{7c}$$

Due to the linearity of Eq. (5), the general integral of this equation can be expressed as

$$f(Y,\tau) = f_h(Y,\tau) + f_p(Y,\tau), \tag{8}$$

where $f_p(Y,\tau)$ is a particular solution of the complete equation and $f_h(Y,\tau)$ is solution of the associated homogeneous equation (6). f_h can be expressed as $f_h(Y,\tau) = \sum_{i=1}^{n} f_i(Y,\tau)$, with $f_i(Y,\tau)$ arbitrary solutions of the associated homogeneous equation.

In some cases a particular integral of the complete equation can be straightforward obtained. In particular, if $g = g(\tau)$ then $f_p(Y,\tau) = f_p(\tau) = \int g(\tau)\, d\tau$. Another interesting case for the present application is $g(Y,\tau) = (\partial f_1/\partial Y)(\partial f_2/\partial Y)$ where f_1 and f_2 are solutions of Eq. (6). Indeed in this case it is easy to verify that

$$f_p(Y,\tau) = -\frac{1}{2} f_1(Y,\tau) f_2(Y,\tau). \tag{9}$$

Specifying respectively with

$$F(Y,s)=\mathcal{L}_\tau\big[f(Y,\tau)\big] \tag{10}$$

and

$$G(Y,s)=\mathcal{L}_\tau\big[g(Y,\tau)\big] \tag{11}$$

the Laplace transforms with respect to time of f and g, we can transform Eqs. (5) and (7) into

$$sF-\frac{\partial^2 F}{\partial Y^2}=G; \tag{12a}$$

$$F(\pm 1,s)=F_w^{\pm}(s), \tag{12b}$$

where $F_w^{\pm}(s)$ are the Laplace transforms of $f_w^{\pm}(\tau)$.

The formal solution of this problem in transformed variables is very simple:

$$F(Y,s)=A(s)\frac{\cosh(\sqrt{s}\,Y)}{\cosh(\sqrt{s})}+B(s)\frac{\sinh(\sqrt{s}\,Y)}{\sinh(\sqrt{s})}+F_p(Y,s), \tag{13}$$

where F_p is a particular integral of the complete equation and $A(s)$ and $B(s)$ are arbitrary functions that are determined imposing the boundary conditions equations (12b).

For the energy equation $F_p(Y,s)$ is difficult to calculate. However, for determining $A(s)$ and $B(s)$ the knowledge of $F_p(Y,s)$ is only necessary on the boundaries.

The explicit, analytical inverse transformation of this equation is not trivial: some results were presented for the momentum equation (1). In [5,9,10] numerically computed inverse transforms were proposed; the authors also showed the possibility to obtain a more general solution by the method of residual. The energy equation was studied by neglecting the effects of dissipation of kinetic energy ($E=0$).

In this paper we present the analytical inverse transform of Eq. (13) for both momentum and energy equations by rapidly converging series expansions taking into account for the effects of the Eckert number. The terms in the series we have obtained are very simple and can be expressed by means of the error functions. Indeed the wanted inverse transformations can be expressed in terms of the Jacobi's $\theta_2(q,\tau)$ and $\theta_3(q,\tau)$ functions. Among the different representations of these functions, useful series expressions (not well-known and more rapidly converging) are the following [11], p. 57:

$$\theta_2(q,\tau)=\frac{1}{\sqrt{\pi\tau}}\sum_{k=-\infty}^{+\infty}(-1)^k e^{-(q+k)^2/\tau}, \tag{14a}$$

$$\theta_3(q,\tau)=\frac{1}{\sqrt{\pi\tau}}\sum_{k=-\infty}^{+\infty}e^{-(q+k)^2/\tau}, \tag{14b}$$

where $0\leqslant q\leqslant 1$. Two important properties of these functions for the present applications are, see [11], p. 107:

$$\mathcal{L}_\tau\left[\frac{\partial\theta_2}{\partial q}(q,\tau)\right]=-2\frac{\cosh[(2q-1)\sqrt{s}]}{\cosh(\sqrt{s})}, \tag{15a}$$

$$\mathcal{L}_\tau\left[\frac{\partial\theta_3}{\partial q}(q,\tau)\right]=2\frac{\sinh[(2q-1)\sqrt{s}]}{\sinh(\sqrt{s})}, \tag{15b}$$

where $0<q<1$.

Then, the inverse transform of Eq. (12a) can be obtained by convolutions of the derivative of the θ_2 and θ_3 Jacobi's functions. These derivatives are:

$$\frac{\partial\theta_2}{\partial q}(q,\tau)=-\frac{2}{\sqrt{\pi}\tau^{3/2}}\left\{q\,e^{-q^2/\tau}+\sum_{k=1}^{+\infty}(-1)^k\big[(k+q)e^{-(k+q)^2/\tau}-(k-q)e^{-(k-q)^2/\tau}\big]\right\}; \tag{16a}$$

$$\frac{\partial\theta_3}{\partial q}(q,\tau)=-\frac{2}{\sqrt{\pi}\tau^{3/2}}\left\{q\,e^{-q^2/\tau}+\sum_{k=1}^{+\infty}\big[(k+q)e^{-(k+q)^2/\tau}-(k-q)e^{-(k-q)^2/\tau}\big]\right\} \tag{16b}$$

and the general solution of Eq. (5) is

$$f(q,\tau)=-\frac{1}{2}\int_0^\tau\frac{\partial\theta_2}{\partial q}(q,\bar\tau)a(\tau-\bar\tau)\,d\bar\tau+\frac{1}{2}\int_0^\tau\frac{\partial\theta_3}{\partial q}(q,\bar\tau)b(\tau-\bar\tau)\,d\bar\tau+f_p(Y,\tau), \tag{17}$$

where $q=(Y+1)/2$, $a(\tau)$ and $b(\tau)$ are the inverse transforms of respectively $A(s)$ and $B(s)$.

The not easy problem of finding a particular integral f_p for the energy equation has been solved using Eq. (9) with appropriate functions f_1 and f_2.

In Appendix A the inverse transforms are derived for power law for $a(\tau)$ and $b(\tau)$.

4. The velocity field

The proposed method of solution is now applied to solve the dynamic field given by Eq. (1) with initial and boundary conditions (2). Specifying respectively with

$$U(Y,s)=\mathcal{L}_\tau\big[u(Y,\tau)\big] \tag{18}$$

and

$$R(s)=\mathcal{L}_\tau\big[r(\tau)\big] \tag{19}$$

the Laplace transforms with respect to time of the unknown velocity field and of the imposed pressure gradient, we can transform Eqs. (1) and (2) into

$$sU-\frac{\partial^2 U}{\partial Y^2}=2\;R(s); \tag{20a}$$

$$U(\pm 1,s)=0. \tag{20b}$$

The solution in transformed variables is

$$U(Y,s)=2\frac{R(s)}{s}\left[1-\frac{\cosh(\sqrt{s}\,Y)}{\cosh(\sqrt{s})}\right]. \tag{21}$$

Hence, by Eq. (17), we have

$$u(q,\tau)=\frac{1}{2}\int_0^\tau\frac{\partial\theta_2}{\partial q}(q,\bar\tau)u_p(\tau-\bar\tau)\,d\bar\tau+u_p(\tau), \tag{22}$$

where $q=(Y+1)/2$ and $u_p(\tau)=\mathcal{L}_\tau^{-1}[2R(s)/s]$.

For power law for the pressure gradient, $r(\tau)=C\tau^n$, it is

$$R(s)=C\frac{n!}{s^{n+1}}, \tag{23}$$

$$u_p(\tau)=2C\frac{\tau^{n+1}}{n+1}; \tag{24}$$

therefore the general solution (22) reduces to

$$u(q,\tau)=\frac{C}{n+1}\left[2\tau^{n+1}+\int_0^\tau\frac{\partial\theta_2}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^{n+1}\,d\bar\tau\right], \tag{25}$$

where $q=(Y+1)/2$. The integral in this equation is analytically calculated in Appendix A.

Due to the linearity of the governing equation (1) the solution for a polynomial law of the pressure gradient can be obtained by linear combinations of power laws. Therefore this method gives the solution for any $r(\tau)$ that can be expanded in Taylor series. In particular in Section 6 the solution for impulsive-start-up flow and linear growth of the pressure gradient are proposed and discussed.

In [12] p. 117, it is proposed the analytical solution of the Navier–Stokes equations for impulsive motion to a constant speed of the upper wall (impulsive Couette flow). The solution was obtained by separation of variables. By present method this solution has been extended to the case of arbitrary time motion of the wall. This solution is reported in Appendix C. Due to the linearity of the problem, this solution can be combined with the unsteady Poiseuille case to obtain the unsteady mixed Poiseuille–Couette flow.

4.1. Analysis of the velocity field

Preliminarily we note that a number of results of physical relevance can be found independently of the assigned pressure gradient law by means of the analysis of the solution in transformed variables by the Tauberian and Abelian theorems (see Appendix B).

4.1.1. Behavior for $\tau \to 0$

The behavior of the solution for small time values can be obtained by the Abelian theorem, looking at the asymptotic behavior ($s \to +\infty$) of the transformed solution. In the case of power law for the pressure gradient we have, for $Y > 0$:

$$s \to +\infty\colon\ U \to 2C\,\frac{n!}{s^{n+2}}\big[1 - \mathrm{e}^{-(1-Y)\sqrt{s}}\big]. \tag{26}$$

The inverse transform of this equation, providing the local behavior for $\tau \to 0^+$ is (see Eq. (B.2)):

$$u(Y,\tau) \approx 2C\tau^{n+1}\left[\frac{1}{n+1} - 2^{2(n+1)} n!\,\mathrm{i}^{2(n+1)}\operatorname{erfc}\left(\frac{1-Y}{2\sqrt{\tau}}\right)\right], \tag{27}$$

where

$$\mathrm{i}^m\operatorname{erfc}(z) = \int_z^{+\infty} \mathrm{i}^{m-1}\operatorname{erfc}(t)\,dt \tag{28}$$

specifies the m repeated integral of the error function, see [13], p. 299 for more details.

Being

$$\mathrm{i}^m\operatorname{erfc}(0) = \frac{1}{2^m\Gamma(m/2+1)}, \qquad \lim_{z\to+\infty} \mathrm{i}^m\operatorname{erfc}(z) = 0, \tag{29}$$

where $\Gamma(z)$ is the gamma function, Eq. (27) shows that, for small time values, the channel flow can be divided in two regions:

(1) *near the wall*: a self-similar *Rayleigh type* layer arises, whose thickness grows as $2\sqrt{\tau}$;
(2) *far from the wall*: there is a *potential* region with the velocity constant across the channel. The *potential* velocity grows as τ^{n+1}.

4.1.2. Asymptotic behavior

The asymptotic behavior of the flow for large time values can be easily obtained by the Tauberian theorem, looking at the behavior for $s \to 0$ of the transformed solution, Eq. (21).

If, for $\tau \to +\infty$, $r(\tau) \to 1$, i.e. the pressure gradient tends asymptotically to a constant value, then the Tauberian theorem ensures that, for $s \to 0^+$, $R(s) \to 1/s$. Therefore, still taking into account for the Tauberian theorem and that, for $s \to 0^+$, $\cosh(\sqrt{s}) \approx 1 + s/2$ and $\cosh(\sqrt{s}Y) \approx 1 + sY^2/2$:

$$\lim_{\tau\to+\infty} u(Y,\tau) = \lim_{s\to 0^+} sU(Y,s) = 1 - Y^2. \tag{30}$$

Hence the unsteady Poiseuille flow tends asymptotically to the steady Poiseuille flow.

4.1.3. Shear stress and mass flow rate

The transformed shear stress at the wall is

$$\left.\frac{\partial U}{\partial Y}\right|_w = 2R(s)\frac{\tanh(\sqrt{s})}{\sqrt{s}}, \tag{31}$$

while the transformed mass flow rate is given by

$$\dot{M}(s) = \int_{-1}^{+1} U(Y,s)\,dY = 4\frac{R(s)}{s}\left[1 - \frac{\tanh(\sqrt{s})}{\sqrt{s}}\right]. \tag{32}$$

Combining equations (32) and (31) we have

$$\dot{M} = 4\frac{R(s)}{s} - \frac{2}{s}\left.\frac{\partial U}{\partial Y}\right|_w, \tag{33}$$

providing a direct relation between mass flow rate and skin friction in an unsteady Poiseuille flow with imposed pressure gradient.

The shear stress at the wall in physical variables is given by the inverse Laplace transform of Eq. (31). From Eq. (B.1), we have:

$$\left.\frac{\partial u}{\partial Y}\right|_w = \frac{2}{\sqrt{\pi}}\int_0^\tau \frac{r(\tau-\bar{\tau})}{\sqrt{\bar{\tau}}}\,d\bar{\tau} + \frac{4}{\sqrt{\pi}}\sum_{k=1}^{+\infty}(-1)^k\int_0^\tau \mathrm{e}^{-k^2/\bar{\tau}}\frac{r(\tau-\bar{\tau})}{\sqrt{\bar{\tau}}}\,d\bar{\tau}. \tag{34}$$

The integrals in this equation can be analytically calculated as shown in Section 6. Similarly, the mass flow rate is related to shear stress obtained by inverse transform of Eq. (33) in this way:

$$\dot{m} = 4\int_0^\tau r(\bar{\tau})\,d\bar{\tau} - 2\int_0^\tau \left.\frac{\partial u(\bar{\tau})}{\partial Y}\right|_w d\bar{\tau}. \tag{35}$$

5. The temperature field

The procedure proposed in Section 3 can be also applied to find the temperature field governed by Eq. (3) with initial and boundary conditions given by Eqs. (4). Due to the linearity of the problem, the analysis can be divided in separated independent problems. In particular, the effect of the dissipation of kinetic energy can be separated from the effect of the assigned wall temperatures. In addition, the latter is given by the sum of a symmetrical field (with respect to the channel axis) and an anti-symmetrical field.

The complete temperature field can be written as follows:

$$\theta(Y,\tau) = \theta_{T_w}(Y,\tau) + \theta_{\Delta T_w}(Y,\tau) + E\ \theta_E(Y,\tau). \tag{36}$$

$\theta_{T_w}(Y,\tau)$ is solution of the symmetrical problem given by the homogeneous heat equation (6) with initial and boundary conditions

$$\theta_{T_w}(Y,0) = 0, \tag{37a}$$

$$\theta_{T_w}(\pm 1,\tau) = \bar{\theta}_w(\tau), \tag{37b}$$

where $\bar{\theta}_w(\tau) = [\theta_w^-(\tau) + \theta_w^+(\tau)]/2$.

$\theta_{\Delta T_w}(Y,\tau)$ is solution of the anti-symmetrical problem given by the homogeneous heat equation with initial and boundary conditions

$$\theta_{\Delta T_w}(Y,0) = 0, \tag{38a}$$

$$\theta_{\Delta T_w}(\pm 1,\tau) = \pm\frac{\Delta\theta_w(\tau)}{2}, \tag{38b}$$

where $\Delta\theta_w(\tau) = \theta_w^+(\tau) - \theta_w^+(\tau)$.

$\theta_{T_w}(Y,\tau)$ and $\theta_{\Delta T_w}(Y,\tau)$ provide the complete temperature field when the effects of the dissipation of kinetic energy are neglected.

$\theta_E(Y,\tau)$, which takes into account for these effects, is solution of the complete equation (3) with $E=1$ and initial and boundary conditions

$$\theta_E(Y,0)=0, \tag{39a}$$

$$\theta_E(\pm 1,\tau)=0. \tag{39b}$$

The combination of these three fundamental solutions provides the temperature field for an arbitrary wall temperature distribution $\theta_w(\tau)$, Prandtl and Eckert number. In the present paper the analytical expression of θ_E has been found explicitly for the case $Pr=1$.

5.1. The case of negligible dissipation of kinetic energy

When the effects of the dissipation of kinetic energy are negligible (zero Eckert number). and assuming $\tau_T=\tau/Pr$ the energy equation (3) reduces to

$$\frac{\partial\theta}{\partial\tau_T}-\frac{\partial^2\theta}{\partial Y^2}=0. \tag{40}$$

The effects of the Prandtl number are taken into account by the scaling of the non-dimensional time τ_T.

5.1.1. Symmetrical field

The solution of the symmetrical problem can be obtained by straightforward application of the procedure shown in Section 3. Denoting with Θ_{T_w} the Laplace transform of θ_{T_w} we have:

$$\Theta_{T_w}(Y,s_T)=\bar\Theta_w(s_T)\frac{\cosh(\sqrt{s_T}\ Y)}{\cosh(\sqrt{(s_T)})}, \tag{41}$$

where $\bar\Theta_w(s_T)$ is the Laplace transform of $\bar\theta_w(\tau_T)$. The inverse transform is

$$\theta_{T_w}(q,\tau_T)=-\frac{1}{2}\int_0^{\tau_T}\frac{\partial\theta_2}{\partial q}(q,\bar\tau_T)\bar\theta_w(\tau_T-\bar\tau_T)\,d\bar\tau_T, \tag{42}$$

where $q=(Y+1)/2$ and $\bar\theta_w(\tau_T-\bar\tau_T)=[\theta_w^+(\tau_T-\bar\tau_T)+\theta_w^-(\tau_T-\bar\tau_T)]/2$. The analytical calculation of the integral in Eq. (42) is proposed in Appendix A in the case of power law for $\theta_w^-(\tau_T)$ and $\theta_w^-(\tau_T)$ and is discussed in Section 6.3.1.

5.1.2. Anti-symmetrical field

In this case the transformed solution is

$$\Theta_{\Delta T_w}(Y,s_T)=\frac{\Delta\Theta_w(s_T)}{2}\frac{\sinh(\sqrt{s_T}\ Y)}{\sinh(\sqrt{(s_T)})}, \tag{43}$$

where $\Delta\Theta_w(s_T)$ is the Laplace transform of $\Delta\theta_w(\tau_T)$. The inverse transform is

$$\theta_{\Delta T_w}(q,\tau_T)=\frac{1}{4}\int_0^{\tau_T}\frac{\partial\theta_3}{\partial q}(q,\bar\tau_T)\Delta\theta_w(\tau_T-\bar\tau_T)\,d\bar\tau_T, \tag{44}$$

where $q=(Y+1)/2$ and $\Delta\theta_w(\tau_T-\bar\tau_T)=\theta_w^+(\tau_T-\bar\tau_T)-\theta_w^-(\tau_T-\bar\tau_T)$.

If $\theta_w^-(\tau_T)$ and $\theta_w^+(\tau_T)$ are power laws, the integral in this equation can be calculated as shown in Appendix A and is discussed in Section 6.3.2.

5.2. The case with dissipation of kinetic energy and $Pr=1$

As we have already noted, the solution of Eq. (3) with initial and boundary conditions (4) including the effect of dissipation of kinetic energy with arbitrary Prandtl number is not trivial. Indeed, nonetheless the linearity of the problem, the Laplace transform of the non-homogeneous term is cumbersome and even more difficult is the inverse transform of a particular solution of the transformed energy equation. The case $Pr=1$ is however very meaningful because the Prandtl number is nearly equal one for most gases.[1] In this case the particular integral of the energy equation can be straightforward obtained in the physical space following the procedure proposed in Section 3.

The general solution of the dynamic field can be expressed as

$$u(Y,\tau)=u_h(Y,\tau)+u_p(\tau), \tag{45}$$

where $u_h(Y,\tau)$ is solution of the homogeneous heat equation (6). Since $\partial u_h/\partial Y=\partial u/\partial Y$, the particular integral θ_p of Eq. (3) with $Pr=1$ can be found in physical variables applying Eq. (9):

$$\theta_p(Y,\tau)=-\frac{E}{2}u_h^2(Y,\tau)=-\frac{E}{2}\big[u(Y,\tau)-u_p(\tau)\big]^2. \tag{46}$$

The contribution $\theta_E(Y,\tau)$ of the dissipation of kinetic energy can be now straightforward obtained and is, in transformed variables:

$$\Theta_E(Y,s)=\frac{4}{s^3}\frac{\cosh(\sqrt{s}\,Y)}{\cosh(\sqrt{s})}+\Theta_p(Y,s,E=1). \tag{47}$$

The inverse Laplace transform of Eq. (47) can be computed as proposed in Section 3:

$$\theta_E(Y,\tau)=\theta_P(Y,\tau,E=1)-\int_0^{\tau}\frac{\partial\theta_2}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^2\,d\bar\tau, \tag{48}$$

where $q=(Y+1)/2$. The calculation of the integral in this equation is proposed in Appendix A and is discussed in Section 6.3.3.

It is interesting to point out that Eq. (48) provides the effect of the Eckert number independently of the temperatures imposed on the walls.

6. Applications

6.1. Impulsive start-up

We assume that at the initial time a jump of the pressure gradient to a finite constant value dp/dx is imposed. This is a special case of the solution proposed in Section 4 with $n=0$ and $C=1$ (the imposed dp/dx is adopted as reference).

With the help of Appendix A the solution is given by:

$$u(q,\tau)=2\tau+u_h(q,\tau), \tag{49}$$

$$u_h(q,\tau)=-h(q,\tau)-\sum_{k=1}^{+\infty}(-1)^k\big[h(k+q,\tau)-h(k-q,\tau)\big], \tag{50}$$

where $q=(Y+1)/2$ and

$$h(a,\tau)=2\big(\tau+2a^2\big)\,\mathrm{erfc}\bigg(\frac{a}{\sqrt{\tau}}\bigg)-\frac{4}{\sqrt{\pi}}a\sqrt{\tau}\,\mathrm{e}^{-a^2/\tau} \tag{51}$$

($\mathrm{erfc}(z)$ specifies the complementary error function).

The velocity profiles for different time values are plotted in Fig. 1. In the plots, 20 terms were adopted in the series of Eq. (49) (but only few terms are sufficient for a good approximation). For small time values the potential and Rayleigh type regions can be noted. The velocity is constant across most of the channel with the variations concentrated near the wall. For large time values the velocity profile tends asymptotically towards the steady Poiseuille solution, also plotted in the figure. For $\tau=2$ the velocity profile is already very near its asymptotic state (the differences between the steady and the unsteady flow are smaller than 1%).

[1] A good approximation of the Prandtl number for a perfect gas is given by the Eucken relationship $Pr=4\gamma/(9\gamma-5)$, where γ is the ratio between the specific heats of the gas, [14], p. 41.

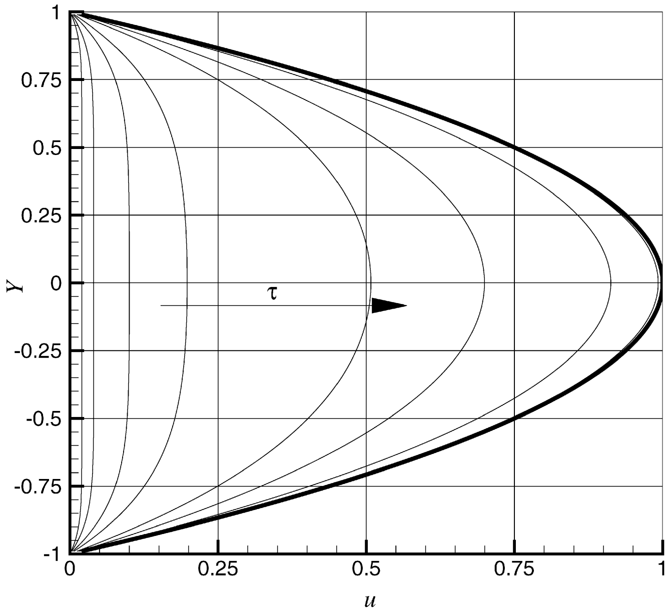

Fig. 1. Impulsive start-up of the Poiseuille flow. Velocity distribution across the channel, $u = u(Y)$, at different times. $\tau = 0.01, 0.02, 0.05, 0.1, 0.3, 0.5, 1, 2$. Thick line: steady Poiseuille solution.

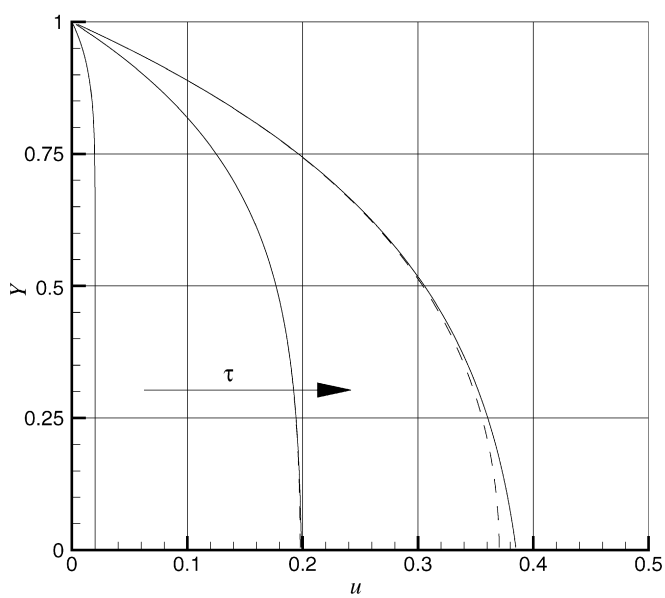

Fig. 2. Impulsive start-up of the Poiseuille flow. Comparison of the local (solid line) and exact (dashed line) solutions for $\tau = 0.01, 0.1, 0.2$.

6.1.1. Behavior for $\tau \to 0$

The local behavior of the solution for $\tau \to 0$, see Eq. (27), is:

$$u(Y,\tau) \approx 2\tau\left[1 - 4\mathrm{i}^2\,\mathrm{erfc}\left(\frac{1-Y}{2\sqrt{\tau}}\right)\right], \tag{52}$$

where

$$\mathrm{i}^2\,\mathrm{erfc}(z) = \frac{1}{4}\left[\left(2z^2+1\right)\mathrm{erfc}(z) - \frac{2}{\sqrt{\pi}}z\,\mathrm{e}^{-z^2}\right]. \tag{53}$$

The relation (52) is compared with the exact expression of the velocity, Eq. (49) in Fig. 2. Only for $\tau \geqslant 0.2$ differences between the two functions can be noted.

6.1.2. Shear stress and mass flow rate

The shear stress at the wall is obtained by Eq. (34) or differentiating Eq. (49):

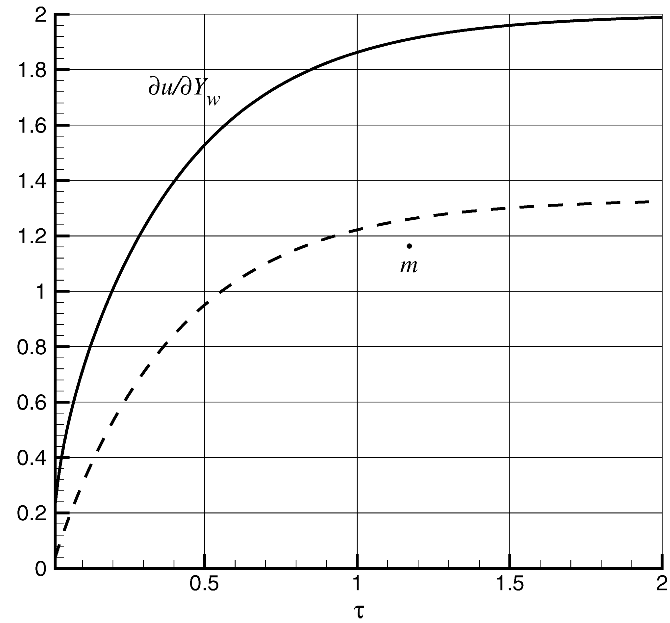

Fig. 3. Impulsive start-up of the Poiseuille flow. Shear stress at the wall (solid line) and mass flow rate (dashed line) versus time.

$$\left.\frac{\partial u}{\partial Y}\right|_w = \frac{4}{\sqrt{\pi}}\left\{\sqrt{\tau} + 2\sum_{k=1}^{+\infty}(-1)^k \times\left[\sqrt{\tau}\,\mathrm{e}^{-k^2/\tau} - \sqrt{\pi}\,k\,\mathrm{erfc}\left(\frac{k}{\sqrt{\tau}}\right)\right]\right\}. \tag{54}$$

The shear stress at the wall and the mass flow rate versus time are plotted in Fig. 3; for large time values they tend asymptotically towards the corresponding values of the steady Poiseuille flow.

6.2. Linear growth of the pressure gradient

In this case $r(\tau) = \tau$. With $C = 1$ and $n = 1$ Eq. (25) reduces to

$$u(q,\tau) = \frac{1}{2}\left\{2\tau^2 - h_2(q,\tau) - \sum_{k=1}^{+\infty}(-1)^k\left[h_2(k+q,\tau) - h_2(k-q,\tau)\right]\right\}, \tag{55}$$

where $q = (Y+1)/2$ and

$$h_2(a,\tau) = \tau^2\phi_0(a,\tau) - 2\tau\phi_1(a,\tau) + \phi_2(a,\tau). \tag{56}$$

The functions $\phi_i(a,\tau)$ are given in Appendix A; in this case:

$$\phi_0(a,\tau) = 2\,\mathrm{erfc}\left(\frac{a}{\sqrt{\tau}}\right), \tag{57a}$$

$$\phi_1(a,\tau) = 2\left[\frac{2}{\sqrt{\pi}}a\sqrt{\tau}\,\mathrm{e}^{-a^2/\tau} - a^2\phi_0(a,\tau)\right], \tag{57b}$$

$$\phi_2(a,\tau) = \frac{2}{3}\left[\frac{2}{\sqrt{\pi}}a\tau^{3/2}\,\mathrm{e}^{-a^2/\tau} - a^2\phi_1(a,\tau)\right]. \tag{57c}$$

The velocity profiles for different time values are plotted in Fig. 4. In this case the maximum velocity indefinitely increases as time grows.

For small time values the velocity profile behaves as

$$u(Y,\tau) \approx \tau^2\left[1 - 32\mathrm{i}^4\,\mathrm{erfc}\left(\frac{1-Y}{2\sqrt{\tau}}\right)\right]. \tag{58}$$

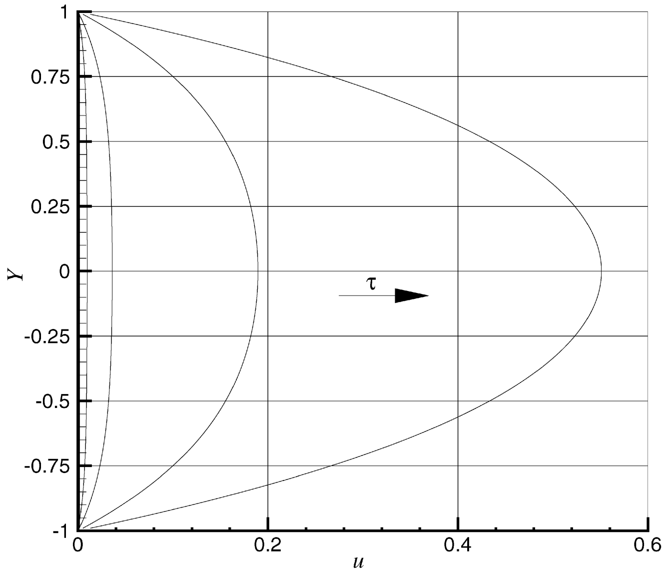

Fig. 4. Poiseuille, flow with linear growth of the pressure gradient. Velocity distribution across the channel, $u = u(Y)$ at different times. $\tau = 0.1$, 0.2, 0.5, 1.

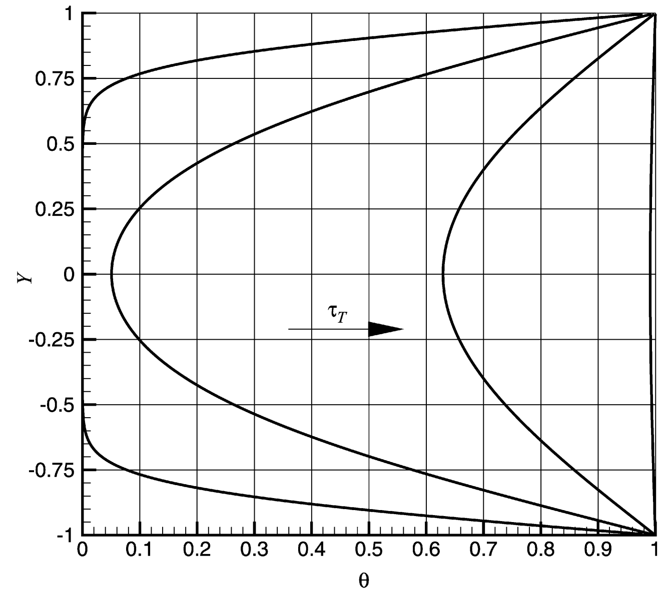

Fig. 5. Impulsive start-up of the Poiseuille flow. Temperature field across the channel, $\theta = \theta(Y)$, at different times. $\tau_T = 0.01$, 0.1, 0.5, 2. $\theta_w^- = \theta_w^+ = 1$, $E = 0$.

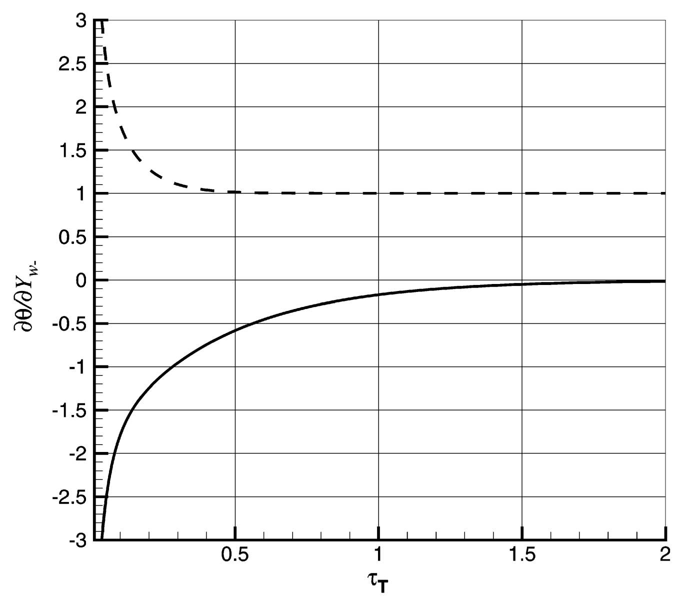

Fig. 6. Impulsive start-up of the Poiseuille flow. Heat fluxes versus time on the lower wall $E = 0$. Solid line: $\theta_w^- = \theta_w^+ = 1$; dashed line: $\theta_w^- = -0.5$, $\theta_w^+ = 0.5$.

6.3. *Temperature field with constant wall temperatures and negligible dissipation of kinetic energy*

We consider an impulsive start-up of the Poiseuille flow assuming that the wall temperatures θ_w^- and θ_w^+ are constant. In this case the solution is a linear function of the average wall temperature $\bar{\theta}_w$ and of wall temperature difference $\Delta\theta_w$.

6.3.1. *Symmetrical problem*

With the help of Appendix A Eq. (42) reduces to

$$\theta_{T_w}(q,\tau_T) = \bar{\theta}_w \left\{ \operatorname{erfc}\left(\frac{q}{\sqrt{\tau_T}}\right) + \sum_{k=1}^{+\infty} (-1)^k \left[\operatorname{erfc}\left(\frac{k+q}{\sqrt{\tau_T}}\right) - \operatorname{erfc}\left(\frac{k-q}{\sqrt{\tau_T}}\right) \right] \right\}, \tag{59}$$

where $q = (Y+1)/2$. This solution is plotted in Fig. 5 for different time values and for $\bar{\theta}_w = 1$. For small time values a *thermal Rayleigh layer* near the wall can be recognized.

The heat fluxes at the walls are

$$\left.\frac{\partial\theta}{\partial Y}\right|_{w\pm} = \pm\frac{\bar{\theta}_w}{\sqrt{\pi\tau_T}} \left[1 + 2\sum_{k=1}^{+\infty} (-1)^k \mathrm{e}^{-k^2/\tau_T} \right]. \tag{60}$$

The heat flux on the lower wall is plotted versus time in Fig. 6 for $\bar{\theta}_w = 1$.

6.3.2. *Anti-symmetrical problem*

Again with the help of Appendix A Eq. (44) gives

$$\theta_{\Delta T_w}(q,\tau_T) = -\frac{\Delta\theta_w}{2} \left\{ \operatorname{erfc}\left(\frac{q}{\sqrt{\tau_T}}\right) + \sum_{k=1}^{+\infty} \left[\operatorname{erfc}\left(\frac{k+q}{\sqrt{\tau_T}}\right) - \operatorname{erfc}\left(\frac{k-q}{\sqrt{\tau_T}}\right) \right] \right\}, \tag{61}$$

where $q = (Y+1)/2$. This solution is plotted in Fig. 7 for different time values and for $\Delta\theta_w = 1$. The heat fluxes at the walls are

$$\left.\frac{\partial\theta}{\partial Y}\right|_{w\pm} = \frac{\Delta\theta_w}{2\sqrt{\pi\tau_T}} \left[1 + 2\sum_{k=1}^{+\infty} \mathrm{e}^{-k^2/\tau_T} \right]. \tag{62}$$

The plot of the heat flux on the lower wall is proposed in Fig. 6 for $\Delta\theta_w = 1$.

6.3.3. *Effect of the Eckert number for Pr = 1 and impulsive start-up*

In this case the contribution of dissipation of kinetic energy to the temperature field is

$$\theta_E(Y,\tau) = \theta_P(Y,\tau,E=1) + h_2(q,\tau) + \sum_{k=1}^{+\infty} (-1)^k \big[h_2(k+q,\tau) - h_2(k-q,\tau) \big], \tag{63}$$

where $q = (Y+1)/2$ and $\theta_p(Y,\tau)$ is given by Eq. (46) adopting the velocity field given by Eq. (49).

In Fig. 8 the temperature profiles θ_E are plotted for different time values with $E = 1$ in this case of impulsive Poiseuille flow. As time grows the temperatures quickly approach the steady solution

$$\theta_E(Y,+\infty) = \frac{1}{3}\left(1 - Y^4\right). \tag{64}$$

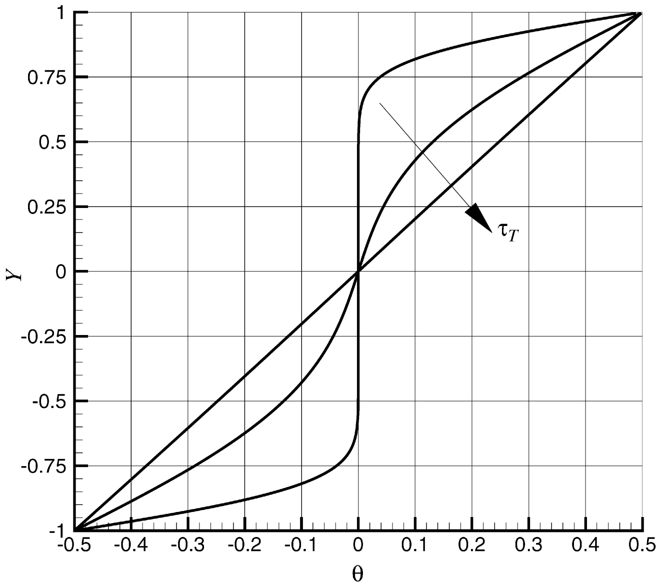

Fig. 7. Impulsive start-up of the Poiseuille flow. Temperature field across the channel, $\theta=\theta(Y)$, at different times. $\tau_T=0.01$, 0.1, 0.5. $\theta_w^-=-0.5$, $\theta_w^+=0.5$, $E=0$.

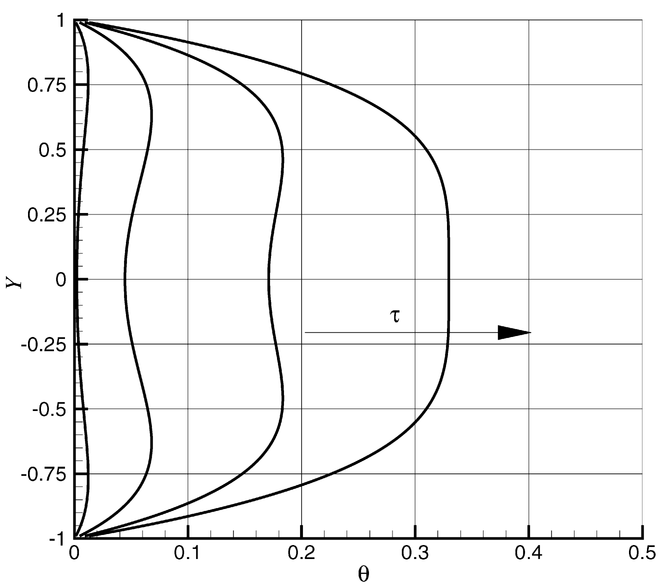

Fig. 8. Impulsive start-up of the Poiseuille flow. Temperature field across the channel for the impulsive start-up of the Poiseuille flow, $\theta=\theta(Y)$, at different times. $\tau=0.2, 0.5, 1, 3$. $E=1$, $\theta_w^-=\theta_w^+=0$.

The associated heat flux at the wall is obtained by differentiating equation (63):

$$\frac{\partial\theta_E}{\partial Y}\bigg|_{w-}=2\tau\frac{\partial u}{\partial Y}\bigg|_{w-}-\frac{16}{3\sqrt{\pi}}\tau^{3/2}+\frac{16}{\sqrt{\pi}}\sum_{k=1}^{+\infty}(-1)^k\left[-\frac{2}{3}\left(\tau^{3/2}+k^2\sqrt{\tau}\right)\mathrm{e}^{-k^2/\tau}+\sqrt{\pi}\left(k\tau+\frac{2}{3}k^3\right)\operatorname{erfc}\left(\frac{k}{\sqrt{\tau}}\right)\right], \tag{65}$$

where $\partial u/\partial Y|_{w-}$ is given by Eq. (54). This relation is plotted in Fig. 9.

Finally, an arbitrary impulsive Poiseuille flow with constant wall temperatures can be obtained by linear combinations of the Eqs. (59), (61) and (63).

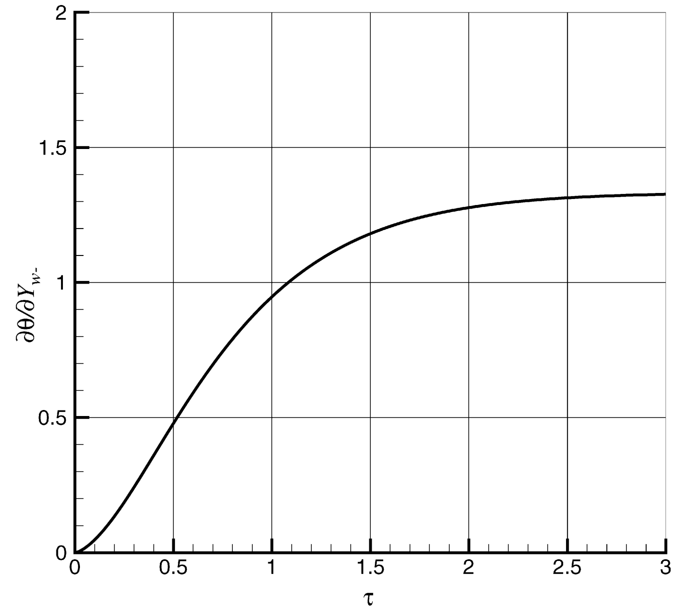

Fig. 9. Impulsive start-up of the Poiseuille flow. Heat flux versus time on the lower wall. $E=1$, $\theta_w^-=\theta_w^+=0$.

7. Conclusions

In this paper a new case has been added to the short list of exact, explicit, analytical solutions of thermo-fluid dynamic fields of practical interest. It is also considered the case of coupled energy and momentum equation by the dissipation of kinetic energy ($E\neq 0$).

It consists in the fully developed laminar, incompressible flow arising in a two-dimensional channel when the imposed time law of the pressure gradient has a power expression (unsteady Poiseuille flow). Due to the linearity of the Navier–Stokes equations in the case of fully developed parallel flows, the solution in the case of arbitrary pressure gradient can also be obtained, provided it can be expanded in Taylor series. The solution has been extended to flow driven by the upper wall motion (unsteady Couette flow). The thermal field has been obtained for arbitrary time law for the wall temperature including the effects of the dissipation of kinetic energy (for Prandtl number equal to one): it was possible to obtain the solution in analytical form because a particular integral of the energy equation was found in a simple form.

The solutions have been obtained by the Laplace transform technique; they can be expressed in terms of a particular form of the Jacobi's θ_2 and θ_3 functions.

The local analysis for small time values, evidences the presence of a Rayleigh type layer near the walls and a potential region far from the walls. Analytical expression of the velocity profiles in the Rayleigh layers and in the potential regions have been proposed. The effects on the thermal field of the Eckert number and wall temperatures are independent and can be separated.

Some cases have been analyzed in detail: impulsive start-up and linear growth of the pressure gradient of the unsteady Poiseuille flow, impulsive variation of the wall temperatures. Simple analytical expressions for the wall shear stress, mass flux and Nusselt number have been obtained and discussed.

Appendix A. The basic inverse transforms

The binomial theorem gives

$$(\tau-\bar{\tau})^{n+1}=\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\tau^{n+1-j}\bar{\tau}^j, \tag{A.1}$$

therefore

$$\int_0^\tau \frac{\partial\theta_2}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^m\,d\bar\tau = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\tau^{m-j}\int_0^\tau \bar\tau^j\frac{\partial\theta_2}{\partial q}(q,\bar\tau)\,d\bar\tau. \tag{A.2}$$

From Eq. (16a) we have

$$\int_0^\tau \bar\tau^j\frac{\partial\theta_2}{\partial q}(q,\bar\tau)\,d\bar\tau = -\frac{2}{\sqrt\pi}\Bigg\{q\int_0^\tau\frac{e^{-q^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau + \sum_{k=1}^{+\infty}(-1)^k\Bigg[(k+q)\int_0^\tau\frac{e^{-(k+q)^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau - (k-q)\int_0^\tau\frac{e^{-(k-q)^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau\Bigg]\Bigg\}. \tag{A.3}$$

The integrals in this equation are all of the same type and can be recursively calculated. Denoting with

$$\phi_j(a,\tau) = \frac{2}{\sqrt\pi}a\int_0^\tau\frac{e^{-a^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau,\quad j=0,1,2,\dots,m, \tag{A.4}$$

where $a=q,k+q,k-q,k=1\dots+\infty$, we have

$$\phi_j(a,\tau) = \frac{2}{2j-1}\left[\frac{2}{\sqrt\pi}a\tau^{(2j-1)/2}e^{-a^2/\tau} - a^2\phi_{j-1}(a,\tau)\right] \tag{A.5}$$

and

$$\phi_0(a,\tau) = 2\,\mathrm{erfc}\left(\frac{a}{\sqrt\tau}\right), \tag{A.6}$$

where $\mathrm{erfc}(z) = 1-2/\sqrt\pi\int_0^z e^{-\zeta^2}\,d\zeta$ is the complementary error function.

In the same way

$$\int_0^\tau \frac{\partial\theta_3}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^n\,d\bar\tau = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\tau^{n-j}\int_0^\tau \bar\tau^j\frac{\partial\theta_3}{\partial q}(q,\bar\tau)\,d\bar\tau. \tag{A.7}$$

Eq. (16b) leads to

$$\int_0^\tau \bar\tau^j\frac{\partial\theta_3}{\partial q}(q,\bar\tau)\,d\bar\tau = -\frac{2}{\sqrt\pi}\Bigg\{q\int_0^\tau\frac{e^{-q^2/\bar\tau}}{\bar\tau^{3/2-j}}d\bar\tau + \sum_{k=1}^{+\infty}\Bigg[(k+q)\int_0^\tau\frac{e^{-(k+q)^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau - (k-q)\int_0^\tau\frac{e^{-(k-q)^2/\bar\tau}}{\bar\tau^{3/2-j}}\,d\bar\tau\Bigg]\Bigg\}. \tag{A.8}$$

The integral in this equation are again of the type specified in Eq. (A.4).

In particular we have:

$$\int_0^\tau \frac{\partial\theta_2}{\partial q}(q,\bar\tau)\,d\bar\tau = -\phi_0(q,\tau) - \sum_{k=1}^{+\infty}(-1)^k\big[\phi_0(k+q,\tau)-\phi_0(k-q,\tau)\big]. \tag{A.9}$$

$$\int_0^\tau \frac{\partial\theta_3}{\partial q}(q,\bar\tau)\,d\bar\tau = -\phi_0(q,\tau) - \sum_{k=1}^{+\infty}\big[\phi_0(k+q,\tau)-\phi_0(k-q,\tau)\big]. \tag{A.10}$$

$$\int_0^\tau \frac{\partial\theta_2}{\partial q}(q,\bar\tau)(\tau-\bar\tau)\,d\bar\tau = -\tau\phi_0(q,\tau)+\phi_1(q,\tau) - \sum_{k=1}^{+\infty}(-1)^k\big[\tau\phi_0(k+q,\tau)-\phi_1(k+q,\tau) - \tau\phi_0(k-q,\tau)+\phi_1(k-q,\tau)\big]. \tag{A.11}$$

$$\int_0^\tau \frac{\partial\theta_3}{\partial q}(q,\bar\tau)(\tau-\bar\tau)\,d\bar\tau = -\tau\phi_0(q,\tau)+\phi_1(q,\tau) - \sum_{k=1}^{+\infty}\big[\tau\phi_0(k+q,\tau)-\phi_1(k+q,\tau) - \tau\phi_0(k-q,\tau)+\phi_1(k-q,\tau)\big]. \tag{A.12}$$

$$\int_0^\tau \frac{\partial\theta_2}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^2\,d\bar\tau = -\tau^2\phi_0(q,\tau)+2\tau\phi_1(q,\tau)-\phi_2(q,\tau) - \sum_{k=1}^{+\infty}(-1)^k\big[\tau^2\phi_0(k+q,\tau)-2\tau\phi_1(k+q,\tau)+\phi_2(k+q,\tau) - \tau^2\phi_0(k-q,\tau)+2\tau\phi_1(k-q,\tau)-\phi_2(k-q,\tau)\big]. \tag{A.13}$$

$$\int_0^\tau \frac{\partial\theta_3}{\partial q}(q,\bar\tau)(\tau-\bar\tau)^2\,d\bar\tau = -\tau^2\phi_0(q,\tau)+2\tau\phi_1(q,\tau)-\phi_2(q,\tau) - \sum_{k=1}^{+\infty}\big[\tau^2\phi_0(k+q,\tau)-2\tau\phi_1(k+q,\tau)+\phi_2(k+q,\tau) - \tau^2\phi_0(k-q,\tau)+2\tau\phi_1(k-q,\tau)-\phi_2(k-q,\tau)\big]. \tag{A.14}$$

Appendix B. Some properties of the Laplace transforms

B.1. Abelian and Tauberian theorems

The initial and asymptotic behaviors of a function $f(t)$ are obtained by analyzing the solution in the transformed space.

Specifying with $F(s)$ the Laplace transform of $f(t)$, the Abelian and Tauberian theorems ensure that

(1) if, for $t\to0^+$, $f(t)\to At^a$ then, for $s\to+\infty$, $F(s)\to A\Gamma(a+1)/s^{a+1}$;
(2) if, for $t\to+\infty$, $f(t)\to At^a$ then, for $s\to0^+$, $F(s)\to A\Gamma(a+1)/s^{a+1}$;

where $\Gamma(x)$ is the gamma function.

B.2. Some inverse Laplace transforms

$$\mathcal{L}_\tau^{-1}\left(\frac{\tanh\sqrt{s}}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi}\sqrt{\tau}}\left[1 + 2\sum_{k=1}^{+\infty}(-1)^k \mathrm{e}^{-k^2/\tau}\right], \tag{B.1}$$

$$\mathcal{L}_\tau^{-1}\left(\frac{\mathrm{e}^{-a\sqrt{s}}}{s^{1+m/2}}\right) = (4\tau)^{m/2}\mathrm{i}^m \operatorname{erfc}\left(\frac{a}{2\sqrt{\tau}}\right), \tag{B.2}$$

where $a \geqslant 0$, $m = 0, 1, 2, 3, \ldots$ and

$$\mathrm{i}^m \operatorname{erfc}(z) = \int_z^{+\infty} \mathrm{i}^{m-1}\operatorname{erfc}(t)\,dt \tag{B.3}$$

is the m repeated integral of the error function.

Appendix C. Unsteady Couette flow

Momentum equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial Y^2} = 0. \tag{C.1}$$

Boundary conditions for the velocity field

$$u(Y, 0) = 0, \tag{C.2a}$$

$$u(-1, \tau) = 0, \tag{C.2b}$$

$$u(+1, \tau) = u_w(\tau), \tag{C.2c}$$

where $u_w(\tau)$ is the assigned velocity of the upper wall referenced to V_{ref}.

Transformed velocity field

$$U(Y, s) = \frac{U_w(s)}{2}\left[\frac{\cosh(\sqrt{s}\ Y)}{\cosh(\sqrt{s})} + \frac{\sinh(\sqrt{s}\ Y)}{\sinh(\sqrt{s})}\right]. \tag{C.3}$$

Velocity field

$$u(q, \tau) = \frac{1}{4}\left\{-\int_0^\tau \frac{\partial\theta_2}{\partial q}(q, \bar\tau)u_w(\tau - \bar\tau)\,d\bar\tau + \int_0^\tau \frac{\partial\theta_3}{\partial q}(q, \bar\tau)u_w(\tau - \bar\tau)\,d\bar\tau\right\}, \tag{C.4}$$

where $q = (Y + 1)/2$.

For power law for the upper wall velocity, $u_w = C\tau^n$, we have

$$u(q, \tau) = \frac{C}{4}\left\{-\int_0^\tau \frac{\partial\theta_2}{\partial q}(q, \bar\tau)(\tau - \bar\tau)^n\,d\bar\tau + \int_0^\tau \frac{\partial\theta_3}{\partial q}(q, \bar\tau)(\tau - \bar\tau)^n\,d\bar\tau\right\}, \tag{C.5}$$

where $q = (Y + 1)/2$.

Asymptotic behavior

If, for $\tau \to +\infty$, $U_w(\tau) \to 1$, i.e. the upper wall velocity tends asymptotically to a constant value, then the Tauberian theorem ensures that, for $s \to 0^+$, $U_w(s) \to 1/s$. Therefore (since $\sinh(\sqrt{s}) \approx \sqrt{s}$ and $\sinh(\sqrt{s}y) \approx \sqrt{s}Y$ for small values of s):

$$\lim_{\tau\to+\infty} u(Y, \tau) = \frac{1}{2}(1 + Y); \tag{C.6}$$

the unsteady Couette flow tends asymptotically to the steady Couette flow.

Behavior for $\tau \to 0$

Again, the local behavior of the solution for $\tau \to 0$ is performed looking at the behavior of the transformed solution, Eq. (C.3) for $s \to +\infty$. For $Y > 0$ we obtain:

$$s \to +\infty:\ U \to U_w(s)\mathrm{e}^{-(1-Y)\sqrt{s}}. \tag{C.7}$$

For power law of the wall velocity ($U_w = Cn!/s^{n+1}$), the velocity in the physical plane for $\tau \to 0$ is

$$u(Y, \tau) \approx 2^{2n}Cn!\tau^n \mathrm{i}^{2n}\operatorname{erfc}\left(\frac{1 - Y}{2\sqrt{\tau}}\right). \tag{C.8}$$

Near the moving wall, for small time values there is a Rayleigh type layer, while the flow is at rest in the remaining part of the channel.

Impulsive start-up ($u_w(\tau) = 1$)

$$u(q, \tau) = -\sum_{k=0}^{+\infty}\left[\operatorname{erfc}\left(\frac{2k + 1 + q}{\sqrt{\tau}}\right) - \operatorname{erfc}\left(\frac{2k + 1 - q}{\sqrt{\tau}}\right)\right], \tag{C.9}$$

where $q = (Y + 1)/2$.

Linear growth of the upper wall ($u_w(\tau) = \tau$)

$$u(q, \tau) = -\frac{1}{2}\sum_{k=0}^{+\infty}\left[h(2k + 1 + q, \tau) - h(2k + 1 - q, \tau)\right], \tag{C.10}$$

where $q = (Y + 1)/2$.

Temperature field with constant wall temperatures and $Pr = 1$ including dissipation of kinetic energy

Only the part of the solution strictly depending on E is different from that of Poiseuille flow; it can be obtained assuming as boundary conditions $\theta(-1, \tau) = \theta(+1, \tau) = 0$.

Particular integral of the energy equation

$$\theta_p(Y, \tau) = \frac{E}{2}u^2(Y, \tau). \tag{C.11}$$

Transformed temperature

$$\Theta(Y, s) = -\frac{E}{4}U_{2w}(s)\left[\frac{\cosh(\sqrt{s}\,Y)}{\cosh(\sqrt{s})} + \frac{\sinh(\sqrt{s}\,Y)}{\sinh(\sqrt{s})}\right] + \Theta_p(Y, s); \tag{C.12}$$

where $U_{2w}(s)$ is the Laplace transform of $u_w^2(\tau)$.

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